Math 279 Lecture 25 Notes

Daniel Raban

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1 Multiplication of Abstract Candidates

1.1 Motivation: Necessity of multiplication in the solution for KPZ

We have formulated a general strategy for treating (subcritical) ill-posed PDEs. Our strategy is to isolate the bad parts, interpret them in an abstract setting, come up with an abstract solution, and use reconstruction theory to give an actual solution. We now would like to describe this strategy in detail for the KPZ equation

$$\begin{cases} h_t = h_{xx} + h_x^2 + \xi - C \\ h(x,0) = h^0(x), \end{cases}$$

where $h : \mathbb{R} \times [0,T] \to \mathbb{R}$ and ξ is white noise. We would like to construct a solution as a fixed point of a suitable operator

$$h = P * (h_x^2 + \xi - C) + P * h,$$

where P is the heat kernel, and by $P * h$, we mean h integrated against P. Last time, we discussed Schauder-type estimates that give regularity for the expression $f \mapsto P * f$ in the sense that if $f \in \mathcal{C}^\alpha_{\text{par}}$, then $P * f \in \mathcal{C}^{\alpha+2}_{\text{par}}$. We argued last time that there would be a multi-layer type Schauder estimate that is applicable for general regularity structure under some natural conditions. We would be able to come up with an operator P such that if $f \in \mathcal{C}^\gamma_M$ ($f : \mathbb{R}^d \to \bigoplus_{\alpha<\gamma} T_\alpha$), then our reconstruction theorem would turn f into $\mathcal{R}f$, which is a distribution that is well approximated by $\Pi_x f(x)$ near x. Moreover,

$$\mathcal{R}(\mathcal{P}f) = P * \mathcal{R}f,$$

and, as we will see later, we can rewrite $\mathcal{P} = \mathscr{I} + \widehat{\mathscr{I}}$, where $\widehat{\mathscr{I}}$ would be a polynomial like dealing with the Taylor approximation of the smooth part of $\mathcal{P} * \mathcal{R}f$. So, in some sense, only the $\mathcal{I}$ part of $\mathcal{P}$ would capture the true nature of the singularity of the kernel $\mathcal{P}$.

To solve this heat kernel equation, we first formulate an abstract variant that can be solved as a fixed point of some nice continuous operator. In other words, the solution we are looking for can be expressed as $h = \mathcal{R}H$, where H solves an equation of the form

$$\begin{aligned} H &= \mathcal{P}((\partial H)^2 + \Xi) + (\mathcal{P} * h^0)\mathbf{1} \\ &= \mathscr{I}((\partial H)^2 + \Xi) + \widehat{\mathscr{I}}((\partial H)^2 + \Xi) + (P * h^0)\mathbf{1}. \end{aligned}$$

Here, Ξ represents ξ in the abstract setting,[1] ∂ represents the spatial derivative (should satisfy $\Pi_a(\partial\tau) = \frac{\partial}{\partial x}(\Pi_a \tau)$), and $(\partial H)^2$ is a candidate for $(\partial H)(\partial H)$.

What do we mean by multiplying two members of our Banach space T? Basically, our regularity structure must be rich enough so that such multiplication can be carried out. Here is our general definition for any multiplication type operation.

Definition 1.1. Given a regularity structure (A, T, G) and two sectors V and $\overline{V}$ (i.e. $V = \bigoplus_{\alpha \in A} V_\alpha$, with V_α a subspace of T_α and with each V_α invariant under G), we say $\star : V \times \overline{V} \to T$ sending $(\tau, \overline{\tau}) \mapsto \tau \star \overline{\tau}$ is a **multiplication** if the following conditions are true:

1. $\star$ is bilinear.
2. If $\tau \in V_\alpha$ and $\overline{\tau} \in \overline{V}_{\alpha'}$, then $\tau \star \overline{\tau} \in T_{\alpha+\alpha'}$.
3. If $\Gamma \in G$, then $\Gamma(\tau \star \overline{\tau}) = (\Gamma\tau) \star (\Gamma\overline{\tau})$.

Example 1.1. Take $\tau = X^k$ and $\overline{\tau} = X^{\overline{k}}$. Then $\tau \star \overline{\tau} = X^{k+\overline{k}}$.

Recall that $f \in \mathcal{C}_M^\gamma$ means that $\|f(x) - \Gamma_{x,y} f(y)\|_\beta \lesssim |x-y|^{\gamma-\beta}$.

Proposition 1.1. *Let $f_1 \in \mathcal{C}_M^\gamma$ with $f_1 : \mathbb{R}^d \to R$ and $f_1(x) \in \bigoplus_{\alpha_1 \le \alpha < \gamma_1} T_\alpha$, and Let $f_2 \in \mathcal{C}_M^\gamma$ with $f_2 : \mathbb{R}^d \to R$ and $f_2(x) \in \bigoplus_{\alpha_2 \le \alpha < \gamma_2} T_\alpha$. Define $(f_1 \star f_2)(x) = f_1(x) \star f_2(x)$. Then $f_1 \star f_2 \in \mathcal{C}_M^\gamma$ with $(\gamma_1 + \alpha_2) \min(\gamma_2 + \alpha_1)$.*

Proof.

$$\begin{aligned} \|\Gamma_{x,y}(f_1 \star f_2)(y) - (f_1 \star f_2)(x)\|_\beta &= \|(\Gamma_{x,y} f_1(y)) \star (\Gamma_{x,y} f_2(y)) - f_1(x) \star f_2(x)\|_\beta \\ &= \|(\Gamma_{x,y} f_1(y) - f_1(x)) * (\Gamma_{x,y} f_2(y) - f_2(x)) \\ &\quad + (\Gamma_{x,y} f_1(y) - f_1(x)) \star f_2(x) \\ &\quad + f_1(x) \star (\Gamma_{x,y} f_2(y) - f_2(x))\| \\ &\lesssim \sum_{\beta_1 + \beta_2 = \beta} (|x-y|^{\gamma_1 - \beta_1} |x-y|^{\gamma_2 - \beta_2} \end{aligned}$$

[1] Professor Rezakhanlou is using Θ in the lectures instead of Ξ because he doesn't like letters with 3 connected components. On a keyboard, I have no such objection.

$$
\begin{aligned}
&\qquad + |x-y|^{\gamma_1-\beta_1} + |x-y|^{\gamma_2-\beta_2}) \\
&= \sum_{\beta_1+\beta_2=\beta} |x-y|^{\gamma_1+\gamma_2-\beta} + |x-y|^{\gamma_1+\beta_2-\beta} \\
&\qquad + |x-y|^{\gamma_2+\beta_1-\beta} \\
&\lesssim |x-y|^{[(\gamma_1+\alpha_2)\wedge(\gamma_2+\alpha_1)]-\beta}.
\end{aligned}
\qquad \square
$$

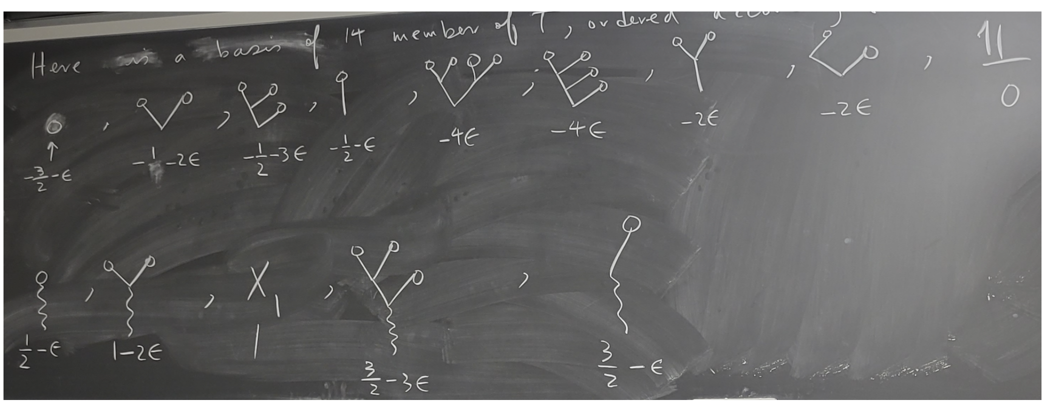
1.2 Basis for a multiplicatively closed regularity structure

We now use our fixed point equation to guess what regularity structure we need. As is done in mathematical physics, we will use graphical notation.[2] We use $\circ$ for Ξ and $\wr$ for the operator $\mathscr{I}$, so that $\mathscr{I}(\Xi)$ is $\wr^\circ$. Because of this, we would also have a component which is like $\mathscr{I}((\partial\wr^\circ)^2)$. This involves $\partial\mathscr{I} =: \mathscr{I}'$. Graphically, we use $|$ for $\mathscr{I}'$ so that $\partial\wr^\circ = |^\circ$.

Here is a table of some of the terms:

degree	expression	model
0	$\mathbf{1}$	constants
$-\frac{3}{2}-\varepsilon$	$\Xi, \circ$	ξ
$\frac{1}{2}-\varepsilon$	$\mathscr{I}(\Xi), \wr^\circ$	$P*\xi$
$-\frac{1}{2}-\varepsilon$	$\mathscr{I}'(\Xi), \vert^\circ$	$(P*\xi)_x$
$-1-2\varepsilon$	$(\mathscr{I}'(\Xi))^2,\ {}^{o}\vee^{o}$	$(P*\xi)_x^2$
$1-2\varepsilon$	$\mathscr{I}((\mathscr{I}'(\Xi))^2),$	$P*(P*\xi)_x^2$
1	X_1	$x-a$

Here is a basis of 14 members of T (ignoring polynomials of higher order), ordered according to their degrees:[3]



[2] I hate this.

[3] There's no way I'm trying to recreate all of these in LaTeX.

This will give us